Model Categories - Mini-Project

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1 Introduction

This mini-project is based upon the axiomatic construction of Model Categories introduced by Quillen in the late 1960s [Quillen, 1967]. This construction of a Model Category allows one to set up the machinery of homotopy theory [Dwyer and Spaliński, 1995] on any category satisfying the Model Category axioms. Whilst most of the language is inherited from Topology, one may equally apply a Model Category structure to purely geometric or algebraic settings. We shall develop the the theory of Model Categories in this mini-project and conclude by providing an example of the Model Category structure on the category of chain complexes over a ring.

2 Preliminaries

Let us briefly set out some notational conventions and basic results. We shall denote a category $\mathbf{C}$ in bold face and denote the category of functors from $\mathbf{D}$ to $\mathbf{C}$ using $\mathbf{C}^{\mathbf{D}}$. We will typically reserve the capital letters X, Y to denote objects in a category, lower case letters f, g to denote morphisms, and upper case F, G to denote functors.

Definition 2.1. (Retract) Let $X, Y \in \mathbf{Obj}(\mathbf{C})$. We say that X is a retract of Y if there are morphisms $r : Y \to X, i : X \to Y$ such that $ri = \mathrm{id}_X$

Lemma 2.1. Let $f, g \in \mathbf{Mor}(\mathbf{C})$ with g an isomorphism. Then if f is a retract of g, f is also an isomorphism.

Proof. Observe that morphisms in the category $\mathbf{Mor}(\mathbf{C})$ are simply commutative squares in $\mathbf{C}$. Thus a retraction gives rise to commutative diagram:

$$\begin{tikzcd}
X_1 \arrow[r, "i_1"] \arrow[d, "f"] & Y_1 \arrow[r, "r_1"] \arrow[d, "g"] & X_1 \arrow[d, "f"] \\
X_2 \arrow[r, "i_2"] & Y_2 \arrow[r, "r_2"] & X_2
\end{tikzcd}$$

And we see that f is invertible with inverse $f^{-1} = r_1 g^{-1} i_2$. $\square$

Definition 2.2. (Lifting)

Given a commutative square of the form below we say that $h : B \to X$ is a lifting of the diagram if the resulting diagram commutes.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r, "g"] \arrow[ur, "h"] & Y \end{tikzcd}$$

We say that i has the left lifting property (LLP) with respect to p, and p has the right lifting property (RLP) with respect to i if for any choice of f, g a lift h exists.

Definition 2.3. ((Co)Base Change)

Given the pushout and pullback diagrams below:

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"] & C \arrow[d, "b'"] \\ B \arrow[r, "a'"] & P \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "g'"] & X \arrow[d, "g"] \\ B \arrow[r, "f"] & C \end{tikzcd}$$

We say that a' is the cobase change of a along b, and that f' is the base change of f along g.

3 Model Category Axioms

In order to set up a general homotopy theory on a category Quillen proposed distinguishing morphisms into three families of maps closed under composition and containing all identity maps: weak equivalences ($\xrightarrow{\sim}$), fibrations ($\twoheadrightarrow$) and cofibrations ($\hookrightarrow$) [Quillen, 1967]. A map that is both a (co)fibration and a weak equivalence is known as an acyclic (co)fibration ($\overset{\sim}{\twoheadrightarrow}, \overset{\sim}{\hookrightarrow}$). If a category $\mathbf{C}$ endowed with the extra data of labelled morphisms then satisfy the axioms set out below, then we have Model Category structure on $\mathbf{C}$.

Definition 3.1. (Model Category Axioms)

- **MC1** Finite limits and colimits exist in $\mathbf{C}$
- **MC2** Suppose f, g, gf are morphisms in $\mathbf{C}$ then if any 2 of the maps are weak equivalence so is the third
- **MC3** Suppose f, g are morphisms in $\mathbf{C}$ with f a retract of g, then f belongs to the same family as g
- **MC4** A commutative square of either of the following forms admits lift:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"] & X \arrow[d, two heads, "p"', "\wr"] \\ B \arrow[r, "g"] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"', "\wr"] & X \arrow[d, two heads, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

- **MC5** Any morphism f may be factored in two ways:

$$A \xrightarrow{f} B \qquad = \qquad A \overset{\sim}{\underset{i}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} B$$

$$A \xrightarrow{f} B \qquad = \qquad A \overset{i}{\hookrightarrow} C \overset{\sim}{\underset{p}{\twoheadrightarrow}} B$$

Note that contrary to [Hovey, 1999] but in line with [Quillen, 1967],[Dwyer and Spaliński, 1995] we adopt the convention of not requiring the factorisations in **MC5** to be functorial. Moreover observe that the existence of finite limits and colimits imply the existence of an initial object ($\emptyset$)and terminal object ($*$) in **C**. Unless stated otherwise let us assume from now on that **C** is endowed with a Model Category Structure. The Model Category Axioms are strong and give rise to a rich theory. Given a non-trivial Model Category structure, the axioms require a lot of work to verify. As such, in some of the examples we present we shall not verify that the axioms hold.

Definition 3.2. An object X in **C** is cofibrant if $\emptyset \to X$ is a cofibration and fibrant if $X \to *$ is a fibration.

It is evident that the axioms for a Model Category are self-dual in the sense that, given a Model Category structure on **C**, then $\mathbf{C}^{\mathrm{op}}$ has a dual Model Category structure given by interchanging the labels of the fibrations and cofibrations [Hovey, 1999]. This duality means that any Model Category theorem has a dual in which one interchanges the place of cofibrations and fibrations.

The following Proposition demonstrates that there is redundancy in distinguishing morphisms into three classes and it would suffice to define only two. Moreover the two parts of the proposition illustrate the duality of the Model Category Axioms.

Proposition 3.1. Let **C** be a Model Category then:

- A map in **C** is a cofibration if and only if it has the LLP with respect to acyclic fibrations
- A map in **C** is a fibration if and only if it has the RLP with respect to acyclic cofibrations

Proof. (Sketch) Let f have the LLP with respect to acyclic fibrations. Apply **MC5** to see that $f = pi$ with i a cofibration. Use **MC4** to see that f is a retract of i and thus by **MC3** f is a cofibration. Apply duality for the second statement. □

Example 3.1. The category **Top** may be given a Model Category structure by defining the weak equivalences to be the weak homotopy equivalences, the fibrations to be the Serre fibrations.

Proposition 3.2. Let **C** be a Model Category then:

- The family of (acyclic) cofibrations is preserved by a cobase change
- The family of (acyclic) fibrations is preserved by a base change

In practice the above Proposition is useful for verifying that the Model Category axioms hold under a proposed Model Category structure.

4 Homotopy Relations

In this section we shall introduce the notion of homotopy relations between maps. We shall define dual notions of left homotopy and right homotopy. Any proofs we provide for results regarding left homotopy shall thus immediately transfer.

Definition 4.1. (Cylinder Object)

A cylinder object for A is an object $A \wedge I$, together with a diagram $A \amalg A \to A \wedge I \xrightarrow{\sim} A$ that factors the map $\mathrm{id}_A + \mathrm{id}_A : A \amalg A \to A$. We say that a cylinder object is good if $A \amalg A \to A \wedge I$ is a cofibration. We say that a cylinder object is very good if it is good and $A \wedge I \xrightarrow{\sim} A$ is also a fibration.

Note that given **MC5** at least one very good cylinder object will always exist. However there is no guarantee of a uniqueness or that the cylinder object may be chosen functorially [Dwyer and Spaliński, 1995]. We denote a cylinder object as $A \wedge I$ to remind us of the topological cylinder $A \times I$.

Definition 4.2. We say maps $f, g : A \to X$ are left homotopic if there exists a cylinder object such that the map $f + g : A \amalg A \to X$ extends to a map $H : A \wedge I \to X$. We say the homotopy is (very) good if the cylinder object is (very) good. We write $f \overset{l}{\sim} g$ if there is a left homotopy from f to g.

A simple application of the Model Category axioms yields the following result.

Lemma 4.1. Suppose that $f \overset{l}{\sim} g : A \to X$. Then there exist a *good* left homotopy between f and g. Moreover if X is fibrant then there exists a *very good* left homotopy between f and g.

Lemma 4.2. If A is cofibrant, then left homotopy $\overset{l}{\sim}$ is an equivalence relation on $\mathrm{Hom}_C(A, X)$.

Proof. (Sketch)
Reflexivity: Let $f : A \to X$, then A is a cylinder object for A and f a left homotopy realising $f \overset{l}{\sim} f$
Symmetry: We may switch the factors of $A \amalg A$.
Transitivity: Let $H : A \wedge I \to X$ and $K : A \wedge I' \to X$ be good left homotopies realising $f \overset{l}{\sim} g$ and $g \overset{l}{\sim} h$. Then we note that $\mathrm{colim}(A \wedge I' \leftarrow A \to A \wedge I)$ is a cylinder object admitting a left homotopy realising $f \overset{l}{\sim} h$. □

If A is not cofibrant we instead consider the equivalence relation generated by $\overset{l}{\sim}$ and denote the set of equivalence classes of $\mathrm{Hom}_C(A, X)$ by $\pi^l(A, X)$.

Lemma 4.3. If A is cofibrant and $p : Y \to X$ is an acyclic fibration, then $p_* : \pi^l(A, Y) \to \pi^l(A, X)$ (induced by post composition with p) is a bijection.

Proof. p_* is well defined since any pair of maps left homotopic in $\pi^l(A, Y)$ via H remain left homotopic under p_* with homotopy pH. Bijectivity follows from **MC4** as p is an acyclic fibration and so the squares below admit lifts:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & Y \arrow[d, two heads, "p"', "\wr"] \\ A \arrow[r, "j"] & X \end{tikzcd} \qquad \begin{tikzcd} A \amalg A \arrow[r, "f+g"] \arrow[d, hook] & X \arrow[d, two heads, "p"', "\wr"] \\ A \wedge I \arrow[r, "H"] & X \end{tikzcd}$$

Let $[j] \in \pi^l(A, X)$ then a lift of the left square is a preimage of $[j]$ hence p_* is surjective. If $[f], [g] \in \pi^l(A, Y)$ and $p_*[f] = p_*[g]$, then let H be a good left homotopy between pf and pg. A lift of the right square is a homotopy between f and g, thus p_* is injective. □

Lemma 4.4. If X is fibrant then we have a well defined map induced by composition in **C**:

$$\pi^l(B, A) \times \pi^l(A, X) \to \pi^l(B, X)$$

Let us quickly summarise the dual notions to left homotopy and the equivalent results, beginning with the dual notion to a cylinder object.

Definition 4.3. (Path Object)
A path object for X is an object X^I, together with a diagram $X \xrightarrow{\sim} X^I \to X \times X$ which factors the diagonal map $(\mathrm{id}_X, \mathrm{id}_X) : X \to X \times X$. We say a path object is good if $X^I \to X \times X$ is a fibration. We say that a path object is very good if it is good and $X \xrightarrow{\sim} X^I$ is also a cofibration.

Definition 4.4. We say maps $f, g : A \to X$ are right homotopic if there exists a path object such that the map $(f, g) : A \to X \times X$ lifts to a map $H : A \to X^I$. We say the homotopy is (very) good if the cylinder object is (very) good. We write $f \overset{r}{\sim} g$ if there is a right homotopy from f to g.

Lemma 4.5. If X is fibrant then $\overset{r}{\sim}$ is an equivalence relation on $\mathrm{Hom}_C(A, X)$.

Lemma 4.6. If X is fibrant and $i : A \to B$ is an acyclic cofibration then $i^* : \pi^r(B, X) \to \pi^l(A, X)$ (induced by pre composition with i) is a bijection.

Lemma 4.7. If A is cofibrant then we have a well defined map induced by composition in bfC:

$$\pi^r(A, X) \times \pi^r(X, Y) \to \pi^r(A, Y)$$

We have dual notions of right and left homotopy. We would like to know how these two notions of homotopy are related.

Lemma 4.8. Let $f, g : A \to X$ be maps. If A is cofibrant and $f \overset{l}{\sim} g$ then $f \overset{r}{\sim} g$. Dually, if X is fibrant and $f \overset{r}{\sim} g$ then $f \overset{l}{\sim} g$.

It is clear that in the case that A is cofibrant and X is fibrant the right and left homotopy equivalence relations on $\mathrm{Hom}_C(A, X)$ coincide. In this case we use $\pi(A, X)$ to denote the set of equivalence classes under the homotopy relation $\sim$. The next lemma shows that in this axiomatic framework weak equivalences between objects which are both fibrant and cofibrant are homotopy equivalences. This is an analogue of the classical Whitehead Theorem in the setting of Model Categories.

Lemma 4.9. Let $f : A \to X$ be maps where A and X are both fibrant and cofibrant. Then f is a weak equivalence if and only if f has a homotopy inverse.

The above Lemma formalises the notion that for sufficiently nice objects our choice of weak equivalence play the role of homotopy equivalences.

5 Homotopy Category

Let us define the Homotopy Category associated to a Model Category. We will give another equivalent formulation of the Homotopy Category that emphasises the notion that our weak equivalences are playing the role of homotopy equivalence.

Definition 5.1. Let X be an object of $\mathbf{C}$ a Model Category. By applying **MC5** to the the map from the initial object $\emptyset \to X$ we can define find cofibrant representative of X, since intermediate object in the factorisation is cofibrant. We denote the acyclic fibration in the factorisation by $p_X : QX \overset{\sim}{\twoheadrightarrow} X$. Dually by factorising the map $X \to *$ we attain a fibrant replacement for X with acyclic cofibration $i_X : X \overset{\sim}{\hookrightarrow} RX$.

Let us choose a (co)fibrant replacement for each object X making the canonical choice whenever X is (co)fibrant.

Lemma 5.1. Given any map $f : X \to Y$ there exist maps $\overline{f} : RX \to RY$ and $\widetilde{f} : QX \to QY$ such that the obvious squares commute.

Moreover these replacement maps may be shown to be unique up to left or right homotopy, and are weak equivalences if and only if the original map is a weak equivalence.

Corollary 5.1. Let $\pi\mathbf{C}_{\text{cofib}}$ denote the category consisting of the cofibrant objects of $\mathbf{C}$ with morphisms the right homotopy classes of maps. Similarly let $\pi\mathbf{C}_{\text{fib}}$ denote the category consisting of fibrant objects of $\mathbf{C}$ with morphisms the left homotopy classes of maps. Then there are well defined replacement functors $Q : \mathbf{C} \to \pi\mathbf{C}_{\text{cofib}}$ and $R : \mathbf{C} \to \pi\mathbf{C}_{\text{fib}}$.

Let $\pi\mathbf{C}$ denote the category of with objects which are those objects that are fibrant and cofibrant in $\mathbf{C}$, and with morphisms given by homotopy classes of maps. The functor R induces a functor $R_{\text{cofib}} : \pi\mathbf{C}_{\text{cofib}} \to \pi\mathbf{C}$.

Definition 5.2. (Homotopy Category)

The homotopy category $\mathbf{Ho}(\mathbf{C})$ of a model category $\mathbf{C}$ is the category with the same objects as $\mathbf{C}$ and morphisms given by $\text{Hom}_{\mathbf{Ho}(\mathbf{C})}(X, Y) = \text{Hom}_{\pi\mathbf{C}}(R_{\text{cofib}}QX, R_{\text{cofib}}QY)$

We could have equally chosen to apply R and a functor induced by Q, and there is no significance as to which we choose as the definition. Let $\gamma : \mathbf{C} \to \mathbf{Ho}(\mathbf{C})$ denote the functor which acts as the identity on objects and the composite $R_{\text{cofib}}Q$ on morphisms.

Example 5.1. If **Top** has the model category structure described earlier then if X a CW complex $\text{Hom}_{\mathbf{Ho}(\mathbf{C})}(X, Y)$ corresponds to the usual homotopy classes of maps.

Theorem 5.1. Let f be a morphism in $\mathbf{C}$. $\gamma(f)$ is an isomorphism in $\mathbf{Ho}(\mathbf{C})$ if and only if f is a weak equivalence. Moreover, any morphism in $\mathbf{Ho}(\mathbf{C})$ can be expressed as the composition morphisms in the image of γ and their inverses.

Despite the obscure construction it turns out that the homotopy category associated to a model category is isomorphic to the localisation with respect to the weak equivalences.

Theorem 5.2. The functor $\gamma : \mathbf{C} \to \mathbf{Ho}(\mathbf{C})$ is a localization of $\mathbf{C}$ with respect to the weak equivalences.

This shows that when defining our Model Category structure it is our choice of the weak equivalences that affect the structure of the corresponding Homotopy Category and the fibrations and cofibrations just play an auxiliary role. The fibrations and cofibrations are not redundant however, as they facilitate the construction of homotopy limits and colimits. Given the brevity of this mini-project we shall not discuss these constructions here.

Let us conclude by giving the classical model category structure on chain complexes presented in [Dwyer and Spaliński, 1995] and [Hovey, 1999]. Let us fix a ring R and denote by **Ch** the category of chain complexes over R with chain maps as morphisms.

Theorem 5.3. (Model Structure on **Ch**)

The following classification of chain maps forms a model category structure on **Ch**. Let $M, N \in$ **Ch** and $f \in \text{Hom}_{\mathbf{Ch}}(M, N)$. Define f to be:

- A weak equivalence if f is a quasi-isomorphism
- A cofibration if f is a degree-wise monomorphism with projective cokernel
- A fibration if f is a degree-wise epimorphism in each positive degree.

Proof. (Sketch)

- **MC1**: The category of modules has all small limits and colimits. Since the small limits and colimits of **Ch** are computed degree-wise then **Ch** has all small limits and colimits.
- **MC2**: If any two of f, g, gf are quasi-isomorphisms then it is obvious the third must be too.
- **MC3**: Verification of this axioms may be considered degree-wise. Diagram chases verify that the retract of a monomorphism, epimorphism or isomorphism remains of the same type. Using the direct summand characteristation of a projective module we see that a retract of a projective module is projective.
- **MC4**: Inductive construction using lifting property of projective modules.
- **MC5**: Explicit construction of intermediate object and maps.

$\square$

References

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